Math 255A' Lecture 19 Notes

Daniel Raban

November 13, 2019

1 Spectral Theorems for Compact Operators

1.1 Spectral theorem for compact, self-adjoint operators

Last time, we proved the following propositions about eigenvalues of compact operators.

Proposition 1.1. *Let* $T \in \mathcal{B}_0(H)$ *and* $\lambda \in \sigma_p(T) \setminus \{0\}$. *Then* $\dim \ker(T - \lambda 1) < \infty$.

Proposition 1.2. *Let* $T \in \mathcal{B}_0(H)$, *and let* $\lambda \neq 0$. *Assume that*

$$\inf\{\|(T-\lambda)h\| : \|h\| = 1\} = 0.$$

Then $\lambda \in \sigma_p(T)$.

Theorem 1.1 (Spectral theorem[1] for self-adjoint operators)**.** *Suppose* T *is comapct and self adjoint. Then*

1. $\sigma_p(T)$ *is countable.*
2. *If* $\sigma_p(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots\}$ *and* P_n *is the projection onto* $\ker(T - \lambda_n)$, *then*
   - $P_n P_m = P_m P_n = 0$ *for all* $m \neq n$ *(i.e.* $\ker(T-\lambda_n) \perp \ker(T - \lambda_m)$*).*
   - $\lambda_n \in \mathbb{R}$ *for all* n.
   - $T = \sum_{n=1}^\infty \lambda_n P_n$ *in* $\|\cdot\|_{\mathrm{op}}$.

The last sum should be thought of as an infinite block diagonal matrix where the blocks are $\lambda_i I_{\operatorname{ran} P_i}$.

Lemma 1.1. *If* T *is normal,* $\ker(T - \lambda) = \ker(T^* - \lambda)$ *is a reducing subspace.*

Proof. If $x \in \ker(T-\lambda)$, then $(T-\lambda)Tx = T(T - \lambda x) = 0$. Then $Tx \in \ker(T - \lambda)$, and same for T^*. □

[1] Tim learned about the spectral theorem at the same time when he was preparing for his driving test. This was a dangerous idea.

Lemma 1.2. *Let T be self-adjoint. If λ, μ are eigenvalues with $\lambda \neq \overline{\mu}$, then* $\ker(T-\lambda) \perp \ker(T-\mu)$.

Proof. Let $x \in \ker(T-\lambda)$ and $y \in \ker(T-\mu)$. Then

$$\lambda \langle x, \mu \rangle = \langle \lambda x, y \rangle = \langle Tx, y \rangle = \langle x, Ty \rangle = \langle x, \mu y \rangle = \overline{\mu} \langle x, y \rangle$$

So $\langle x, y \rangle = 0$. $\square$

Lemma 1.3. *If T is self-adjoint, then $\sigma_p(T) \subseteq \mathbb{R}$.*

Proof. If $x \in \ker(T-\lambda) \setminus \{0\}$, then

$$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\lambda} \langle x, x \rangle ,$$

so $\lambda = \overline{\lambda}$. $\square$

Lemma 1.4. *Let T be compact and self-adjoint. Then at least one of $\|T\|_{\text{op}}, -\|T\|_{\text{op}} \in \sigma_p(T)$.*

Proof. Recall that

$$\|T\| = \sup\{ |\langle Tx, x \rangle| : \|x\| = 1 \}.$$

Since $\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}$, we will assume that this equals $\sup\{\langle Tx, x \rangle : \|x\| = 1\}$ (the other case is the negative case). Suppose $\|x_n\| = 1$ and $\langle Tx_n, x_n \rangle \to 1$. Then

$$\|Tx_n - \lambda x_n\|^2 = \underbrace{\langle Tx_n, Tx_n \rangle}_{\leq \lambda^2} - \underbrace{2\lambda \langle Tx_n, x - N \rangle}_{\to -2\lambda^2} + \lambda^2 \|x_n\|^2.$$

By the lemma, $\lambda \in \sigma_p(X)$. $\square$

Proof. If $\|T\| \in \sigma_p(T)$, let $\lambda_1 = \|T\|$, and let P_1 be the projection onto $\ker(T-\lambda_1)$ (this is reducing). Now consider $T_1 = T|_{\ker(T-\lambda_1)^\perp}$. This is compact, self-adjoint, and $\|T_1\| \leq \|T\|$. If $-\|T\| \in \sigma_p(T_1)$, let $\lambda_2 = -\|T\|$ and $P_2 = P_{\ker(T-\lambda_2)}$. Then let $T_2 := T(1-P_1)(1-P_2) = T|_{(\ker(T-\lambda_1)+\ker(T-\lambda_2))^{\backslash perp}}$. Now $\|T_2\| < \|T\|$.

Continue to produce a sequence of eigenvalues $\|lambda_3, \lambda_4, \lambda_5, \ldots$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots$ and a sequence of projections P_i onto $\ker(T-\lambda_i)$. In this sequence of eigenvalues, there are no consecutive equalities. Also, we have $|\lambda_{i+1}| = \|T_{i+1}\|$ and $T_{i+1} := T(1-P_1)\cdots(1-P_i)$.

Next, we show that $|\lambda_i| \to 0$. If not, let $x_i \in \ker(T-\lambda_i)$ be such that $\|x_I\| = 1$. Then $Tx_i = \lambda_i x_i$ is a sequence of orthogonal vectors not going to 0, contradicting compactness.

Now consider $S = \sum_{i=1}^\infty \lambda_i P_i$. We want to show that $S = T$. Call $S_N = \sum_{i=1}^N \lambda_i P_i$. We have by Parseval's theorem that

$$\|(S - S_N)x\|^2 = \left\| \sum_{i=N+1}^\infty \lambda_i P_i x \right\|^2 = \sum_{i=N+1}^\infty |\lambda|^2 \|P_i x\|^2 \leq |\lambda_{N+1}| \sum_{i=N+1}^\infty \|P_i x\|^2 \to 0.$$

Now

$$(T - S_N)x = (T - S_N)x_1 + (T - S_N)x_2$$

where $x_1 = (P_1 + \cdots + P_N)x, x_2 = x - x_1 \perp \operatorname{span}(\ker(T - \lambda_1), \ldots, \ker(T - \lambda_N)$. Now split $x_1 = P_1 x_1 + \cdots + P_N x_1 = x_{1,1} + \cdots + x_{1,N}$ to get

$$\begin{aligned} &= \sum_{i=1}^{N} (T - S_N)x_{1,i} + (T - S_n)X_2 \\ &= Tx_2. \end{aligned}$$

And we also have

$$\|Tx_2\| = \|T_{N+1}x_2\| \leq |\lambda_{N+1}|\|x_2\| \leq |\lambda_{N+1}|\|x\| \to 0.$$

Finally, we have enumerated all the eigenvalues, so there are only countably many. $\square$

The proof gives us the following facts, as well.

Corollary 1.1. *Let T be compact and self-adjoint.*

1. *The P_n each have finite rank.*
2. $|\lambda_n| \to 0$.
3. $\ker T = (\sum_n \operatorname{ran} P_n)^\perp$

Here is a formulation which makes this look even more like diagonalization:

Corollary 1.2. *There exist an orthonormal basis $(e_n)_n$ for $(\ker T)^\perp$ and $(\mu_n)_n$ in $\mathbb{R}$ with $\mu_n \to 0$ such that*

$$Tx = \sum_n \mu_n \langle x, e_n \rangle e_n, \qquad \forall x \in H.$$

Proof. Let $T = \sum_m \lambda_m P_m$. Convert to the above form. Each λ_m appears $\dim P_m$-many times as a μ_m. $\square$

1.2 Spectral theorem for compact, normal operators

If N is normal, then $N = S + iT$, where S, T are self-adjoint and $ST = TS$. T and S are linear combinations of N and N^*, so if N is comapct, so are S, T.

Proposition 1.3. *Suppose $S = \sum_{i=1}^\infty \alpha_i P_i$ with $\alpha_i \in \mathbb{F}$ distinct (and nonzero) and P_i orthogonal projections, If $ST = TS$, then $P_i T P_i = TP_i$ for all i. If S is self-adjoint, then $P_i T = TP_i$ for all i.*

Proof. Check that $\ker(S-\alpha_i) = \operatorname{ran} P_i$. If $Sx = \alpha_i x$, then $STx = TSx = T(\alpha_i x) = \alpha_i Tx$. This shows that $P_i T P_i = T P_i$.

If $S = S^*$, then P_i reduces T for all i:

$$ST^* = S^*T^* = (TS)^* = (ST)^* = T^*S^* = T^*S.$$

So $P_i T = T P_i$. □

Theorem 1.2 (Spectral theorem for compact, normal operators)**.** *Let N be comapct and normal. Then*

1. *$\sigma_p(T)$ is countable.*
2. *If $\sigma_p(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots\}$ and P_n is the projection onto $\ker(T-\lambda_n)$, then*
   - *$P_n P_m = P_m P_n = 0$ for all $m \neq n$ (i.e. $\ker(N-\lambda_n) \perp \ker(N-\lambda_m)$).*
   - *$N = \sum_{n=1}^\infty \lambda_n P_n$ in $\|\cdot\|_{\mathrm{op}}$.*

Proof. Let $N = S + iT$ with S, T self-adjoint, and write $S = \sum_{k\geq 1} \lambda_k^S P_k^S$. Now P_k^S reduces T for all k. Now choose a further decomposition $P_k^S = Q_{k,1} + \cdots + Q_{k,m_k}$ such that $TP_k^S = TQ_{k,1} + \cdots + TQ_{k,m_k} = \beta_{k,1}^T Q_{k,1} + \cdots + \beta_{k,m_k}^T Q_{k,m_k}$. Now $S = \sum_k \sum_{i=1}^{m_k} \lambda_k^S Q_{k,i}$, and $T = \sum_k \sum_{i=1}^{m_k} \beta_{k,1} Q_{k,i}$. So

$$S + iT = \sum_k \sum_{i=1}^{m_k} (\lambda_k^S + i\beta_{k,i}) Q_{k,i}.$$

Check that $\beta_{k,i} \to 0$ and that $Q_{k,i} Q_{\ell,j} = Q_{\ell,j} Q_{k,i} = 0$. □

For non-compact operators, we will have an analogous result that gives $T = \int_a^b \lambda \, dE(\lambda)$. We have to make sense of this integral.